

# ON PLURICANONICAL LOCALLY CONFORMALLY KÄHLER MANIFOLDS

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**ABSTRACT.** We give a short proof of the fact that compact pluricanonical locally conformally Kähler manifolds have parallel Lee form.

A locally conformally Kähler (lcK) manifold is a complex manifold  $(M, J)$  together with a Hermitian metric  $g$  which is conformal to a Kähler metric in the neighbourhood of every point. The logarithmic differentials of the conformal factors glue up to a globally defined closed 1-form  $\theta$ , called the *Lee form*, such that the fundamental 2-form  $\Omega := g(J\cdot, \cdot)$  satisfies

$$(1) \quad d\Omega = 2\theta \wedge \Omega.$$

When  $\theta$  is parallel with respect to the Levi-Civita connection  $\nabla$  of  $g$ , the lcK manifold  $(M, J, g)$  is called *Vaisman*. G. Kokarev introduced in the context of harmonic maps [1] the seemingly larger class of *pluricanonical lcK manifolds*, defined as those lcK manifolds  $(M, g, J)$  for which the covariant derivative of the metric dual  $\xi := \theta^\sharp$  of the Lee form anti-commutes with the complex structure  $J$ :

$$(2) \quad \nabla_{JX}\xi = -J\nabla_X\xi, \quad \forall X \in TM.$$

In their recent preprint [3], L. Ornea and M. Verbitsky prove the following result using rather involved arguments:

**Theorem.** *Every compact pluricanonical lcK manifold  $(M, J, g)$  is Vaisman.*

We give in the present note an alternate direct proof of this result.

It is well known that on Hermitian manifolds, the exterior derivative of the fundamental 2-form determines its covariant derivative. The formula for the covariant derivative of  $J$  determined by (1) is (see e.g. [2]):

$$(3) \quad \nabla_X J = X \wedge J\theta + JX \wedge \theta, \quad \forall X \in TM,$$

where if  $\alpha$  is a 1-form,  $J\alpha$  denotes the 1-form defined by  $(J\alpha)(Y) := -\alpha(JY)$  for all tangent vectors  $Y$ , and  $X \wedge \alpha$  is the endomorphism of the tangent bundle defined by

$$(X \wedge \alpha)(Y) := g(X, Y)\alpha^\sharp - \alpha(Y)X.$$

Consider the symmetric endomorphism  $S := \nabla\xi$ . We need to show that, under the compactness assumption, (2) implies the vanishing of  $S$ . By (3) we have

$$(4) \quad \nabla_X\xi = SX, \quad \nabla_X(J\xi) = JSX + \theta(X)J\xi + \theta(JX)\xi - |\theta|^2 JX,$$

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which by lowering the indices also reads

$$(5) \quad \nabla_X \theta = (SX)^\flat, \quad \nabla_X (J\theta) = (JSX)^\flat + X \lrcorner (\theta \wedge J\theta - |\theta|^2 \Omega)$$

Since  $SJ = -JS$  by (2), the endomorphism  $JS$  is symmetric. From (5) we thus get

$$(6) \quad d(J\theta) = 2(\theta \wedge J\theta - |\theta|^2 \Omega),$$

$$(7) \quad \mathcal{L}_\xi g = 2g(S\cdot, \cdot), \quad \mathcal{L}_{J\xi} g = 2g(JS\cdot, \cdot).$$

Taking a further exterior derivative in (6) and using (1) yields

$$0 = d^2(J\theta) = 2(-\theta \wedge d(J\theta) - d(|\theta|^2) \wedge \Omega - |\theta|^2 d\Omega) = -2d(|\theta|^2) \wedge \Omega,$$

whence  $|\theta|^2$  is constant on  $M$ . We thus obtain for every tangent vector  $X$

$$0 = X(|\xi|^2) = 2g(\nabla_X \xi, \xi) = 2g(SX, \xi) = 2g(S\xi, X),$$

showing that  $S\xi = 0$  (and therefore also  $SJ\xi = 0$  using (2)). From (4) we thus get  $\nabla_{J\xi} \xi = \nabla_\xi (J\xi) = 0$ , so in particular

$$(8) \quad [\xi, J\xi] = 0.$$

Cartan's formula shows that on every lcK manifold

$$(9) \quad \mathcal{L}_{J\xi} \Omega = d(J\xi \lrcorner \Omega) + J\xi \lrcorner d\Omega = -d\theta + 2J\xi \lrcorner (\theta \wedge \Omega) = 0.$$

Moreover, on pluricanonical manifolds, equation (6) gives

$$(10) \quad \mathcal{L}_\xi \Omega = d(\xi \lrcorner \Omega) + \xi \lrcorner d\Omega = d(J\theta) + 2\xi \lrcorner (\theta \wedge \Omega) = 0.$$

From (7) and (10) we infer

$$(11) \quad \mathcal{L}_\xi J = 2JS, \quad \mathcal{L}_{J\xi} J = -2S.$$

We notice that (8) implies  $[\mathcal{L}_\xi, \mathcal{L}_{J\xi}] = \mathcal{L}_{[\xi, J\xi]} = 0$ , and thus from (11):

$$(12) \quad \mathcal{L}_\xi S = -\frac{1}{2} \mathcal{L}_\xi \mathcal{L}_{J\xi} J = -\frac{1}{2} \mathcal{L}_{J\xi} \mathcal{L}_\xi J = -\mathcal{L}_{J\xi} (JS) = 2S^2 - J\mathcal{L}_{J\xi} S,$$

which (after composing with  $J$  on the left) also reads

$$(13) \quad \mathcal{L}_{J\xi} S = J\mathcal{L}_\xi S - 2JS^2.$$

Taking a further Lie derivative in (12) and using (11) yields

$$\begin{aligned} \mathcal{L}_{J\xi} \mathcal{L}_\xi S &= 2S\mathcal{L}_{J\xi} S + 2(\mathcal{L}_{J\xi} S)S + 2S\mathcal{L}_{J\xi} S - J\mathcal{L}_{J\xi}^2 S \\ &= 4S\mathcal{L}_{J\xi} S + 2(\mathcal{L}_{J\xi} S)S - J\mathcal{L}_{J\xi}^2 S. \end{aligned}$$

Similarly, from (13) and (11) we obtain:

$$\begin{aligned} \mathcal{L}_\xi \mathcal{L}_{J\xi} S &= 2JS\mathcal{L}_\xi S + J\mathcal{L}_\xi^2 S - 4JS^3 - 2J(\mathcal{L}_\xi S)S - 2JS\mathcal{L}_\xi S \\ &= J\mathcal{L}_\xi^2 S - 4JS^3 - 2J(\mathcal{L}_\xi S)S \\ &= J\mathcal{L}_\xi^2 S - 8JS^3 - 2(\mathcal{L}_{J\xi} S)S. \end{aligned}$$

Comparing the last two equations and using  $\mathcal{L}_\xi \mathcal{L}_{J\xi} = \mathcal{L}_{J\xi} \mathcal{L}_\xi$  we obtain

$$(14) \quad J(\mathcal{L}_\xi^2 S + \mathcal{L}_{J\xi}^2 S) = 4S\mathcal{L}_{J\xi} S + 4(\mathcal{L}_{J\xi} S)S + 8JS^3.$$

We compose with  $-SJ$  to the left and take the trace in the above equation:

$$\mathrm{tr}(S(\mathcal{L}_\xi^2 S + \mathcal{L}_{J\xi}^2 S)) = -4\mathrm{tr}(SJS(\mathcal{L}_{J\xi} S)) - 4\mathrm{tr}(SJ(\mathcal{L}_{J\xi} S)S) + 8\mathrm{tr}(S^4) = 8\mathrm{tr}(S^4),$$

from the trace identity and the hypothesis  $SJ = -JS$ . Using this we compute:

$$\begin{aligned} (\mathcal{L}_\xi^2 + \mathcal{L}_{J\xi}^2)(\mathrm{tr}(S^2)) &= \mathrm{tr}((\mathcal{L}_\xi^2 S)S + 2(\mathcal{L}_\xi S)^2 + S(\mathcal{L}_\xi^2 S) + (\mathcal{L}_{J\xi}^2 S)S + 2(\mathcal{L}_{J\xi} S)^2 + S(\mathcal{L}_{J\xi}^2 S)) \\ &= 2\mathrm{tr}((\mathcal{L}_\xi S)^2) + 2\mathrm{tr}((\mathcal{L}_{J\xi} S)^2) + 2\mathrm{tr}(S(\mathcal{L}_\xi^2 S) + S(\mathcal{L}_{J\xi}^2 S)) \\ &= 2\mathrm{tr}((\mathcal{L}_\xi S)^2) + 2\mathrm{tr}((\mathcal{L}_{J\xi} S)^2) + 16\mathrm{tr}(S^4). \end{aligned}$$

By taking the Lie derivative of the relation  $g(S\cdot, \cdot) = g(\cdot, S\cdot)$  with respect to  $\xi$  and using (7) we immediately get  $g(\mathcal{L}_\xi S\cdot, \cdot) = g(\cdot, \mathcal{L}_\xi S\cdot)$ , i.e., the endomorphism  $\mathcal{L}_\xi S$  is symmetric. Taking now the Lie derivative of the relation  $SJ + JS = 0$  with respect to  $\xi$  and using (12) we obtain that  $\mathcal{L}_\xi S$  anti-commutes with  $J$ . Finally, (13) shows that the symmetric part of  $\mathcal{L}_{J\xi} S$  is  $J\mathcal{L}_\xi S$  and its skew-symmetric part is  $-2JS^2$ . The previous relation thus reads

$$\begin{aligned} (\mathcal{L}_\xi^2 + \mathcal{L}_{J\xi}^2)(\mathrm{tr}(S^2)) &= 2\mathrm{tr}((\mathcal{L}_\xi S)^2) + 2\mathrm{tr}((\mathcal{L}_{J\xi} S)^2) + 16\mathrm{tr}(S^4) \\ &= 2\mathrm{tr}((\mathcal{L}_\xi S)^2) + 2\mathrm{tr}((\mathcal{L}_\xi S)^2) - 4\mathrm{tr}(S^4) + 16\mathrm{tr}(S^4) \\ &= 4\mathrm{tr}((\mathcal{L}_\xi S)^2) + 8\mathrm{tr}(S^4). \end{aligned}$$

We use now the compactness assumption: there exists a point  $x_{\max} \in M$  where  $\mathrm{tr}(S^2)$ , the square norm of  $S$ , attains its supremum. At  $x_{\max}$  the left hand side of the equation above is non-positive, while the right hand side is non-negative (since we have seen that  $\mathcal{L}_\xi S$  is symmetric). We deduce that  $\mathrm{tr}(S^4)$  – and thus  $S$  itself – both vanish at  $x_{\max}$ , so  $S$  vanishes identically. This is the conclusion of the Theorem.

## REFERENCES

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